

GENERALIZED d -KOSZUL MODULES

NING BIAN, YU YE, PU ZHANG*

ABSTRACT. Generalized d -Koszul modules are introduced to solve an open problem: the odd Ext-module $E^{\text{odd}}(M)$ of a d -Koszul module M over a d -Koszul algebra Λ is a Koszul module over the even Yoneda algebra $E^{\text{ev}}(\Lambda)$.

Introduction

For an integer $d \geq 2$, a d -Koszul algebra was introduced and studied by R. Berger [B1], and developed by E. L. Green et al. [GMMZ] to the nonlocal case. If $d = 2$ it is the usual Koszul algebra. This class of generalized Koszul structures turns out to be important for example in theory of the Artin-Shelter algebras, the Calabi-Yau algebras, and the Yang-Mills algebras (see e.g. [B1], [B2], [CD]).

Let Λ be a d -Koszul algebra and M a d -Koszul Λ -module. It was shown in Theorem 6.1 of [GMMZ] that the even Ext-algebra $E^{\text{ev}}(\Lambda)$ is a Koszul algebra and the even Ext-module $E^{\text{ev}}(M)$ is a Koszul $E^{\text{ev}}(\Lambda)$ -module. This generalizes the corresponding result of J. Backelin and R. Fröberg [BF] on the local Koszul algebras. An open problem was raised by E. L. Green et al. [GMMZ], Section 6: Is the odd Ext-module $E^{\text{odd}}(M)$ also a Koszul module over $E^{\text{ev}}(\Lambda)$? E. N. Marcos and R. Martínez-Villa [MM] proved that this is the case if the orthogonal algebra Λ^\perp is also a d -Koszul algebra. However, in general Λ^\perp is not a d -Koszul algebra (see [B1]; also Example 2 in [MM]). So the problem remains to be open.

In this paper we introduce the so-called generalized d -Koszul modules. This is a natural class of graded modules. For example, the syzygies of a d -Koszul module are generalized d -Koszul modules up to shifts. Also for each i , $J^i M$ is a generalized d -Koszul module up to shift, where M is a generalized d -Koszul module over a d -Koszul algebra, and J is the graded Jacobson radical.

Our main result is as follows.

Main Theorem. *Let Λ be a d -Koszul algebra, and M a generalized d -Koszul Λ -module. Then $E^{\text{ev}}(M)$ is a Koszul module over the Koszul algebra $E^{\text{ev}}(\Lambda)$.*

As a consequence, we have

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*The corresponding author.

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mathbn@hotmail.com, yeyu@ustc.edu.cn, pzhang@sjtu.edu.cn.

Corollary. *Let Λ be a d -Koszul algebra, and M a d -Koszul Λ -module. Then $E^{\text{odd}}(M)$ is a Koszul module over the Koszul algebra $E^{\text{ev}}(\Lambda)$.*

This answers in the affirmative the open problem mentioned above.

1. Preliminaries

We fix the notations and recall some facts frequently used later. For the details we refer to [BGS], [GM], and [GMMZ].

1.1. Throughout Λ is a *standardly graded algebra* over a field k (see [GM], p.250), i.e., $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \cdots$ is a positively graded k -algebra satisfying the following three conditions:

- (i) $\Lambda_0 = k^r$ for some integer $r \geq 1$,
- (ii) $\dim_k \Lambda_i < \infty$, $\forall i \geq 0$,
- (iii) $\Lambda_i \Lambda_j = \Lambda_{i+j}$, $\forall i, j \geq 0$.

A left *graded Λ -module* M is a Λ -module together with a decomposition of k -spaces $M = \bigoplus_{i \in \mathbb{Z}} M_i$ such that $\Lambda_i M_j \subseteq M_{i+j}$, $\forall i, j \in \mathbb{Z}$. Let M and N be graded Λ -modules. A Λ -homomorphism $f : M \rightarrow N$ is a *graded homomorphism* if $f(M_i) \subseteq N_i$, $\forall i \in \mathbb{Z}$. For $M \in \text{Gr}(\Lambda)$, let $M[n]$ denote the graded module with $M[n]_i = M_{i-n}$. Let $\Lambda\text{-Mod}$ be the category of the left Λ -modules, $\text{Gr}(\Lambda)$ the category of the left graded Λ -modules and graded homomorphisms, and $\text{gr}(\Lambda)$ the full subcategory of $\text{Gr}(\Lambda)$ consisting of finitely generated Λ -modules. Then $\Lambda\text{-Mod}$ and $\text{Gr}(\Lambda)$ are abelian categories; and $\text{gr}(\Lambda)$ is abelian if Λ is noetherian. Let $\text{Hom}_{\text{Gr}(\Lambda)}$ and $\text{Ext}_{\text{Gr}(\Lambda)}^i$ denote the homomorphisms and extensions in $\text{Gr}(\Lambda)$, as opposed to the usual Hom_Λ and Ext_Λ^i in $\Lambda\text{-Mod}$.

Let I be a subset of \mathbb{Z} , and $M \in \text{Gr}(\Lambda)$. M is *generated in degrees in I* , if $M = \Lambda(\bigoplus_{j \in I} M_j)$; M is *generated in degree i* if $M = \Lambda M_i$; M is *supported above degree n* if $M_j = 0$ for $j < n$; and M is *concentrated in degrees in I* if $M_i = 0$ for $i \notin I$.

Let J be the ideal $\bigoplus_{i \geq 1} \Lambda_i$ of Λ . The trivial Λ -module Λ_0 is the lift of the Λ_0 -module Λ_0 via the k -algebra homomorphism $\Lambda \rightarrow \Lambda/J = \Lambda_0$. It is a graded Λ -module concentrated in degree 0. We need the following well-known fact.

Lemma 1.1. *Let $M \in \text{Gr}(\Lambda)$, and I be a subset of \mathbb{Z} . If $\text{Hom}_{\text{Gr}(\Lambda)}(M, \Lambda_0[j]) = 0$ for $j \notin I$, then M is generated in degrees in I .*

Proof. For the convenience of the reader we include a justification. Put $L := M/\Lambda(\bigoplus_{i \in I} M_i)$. If $L \neq 0$, then $L/JL \neq 0$. While L/JL is a graded module over the semisimple algebra Λ_0 , it follows that $\text{Hom}_{\text{Gr}(\Lambda)}(L/JL, \Lambda_0[j]) \neq 0$ for some $j \notin I$, and hence $\text{Hom}_{\text{Gr}(\Lambda)}(M, \Lambda_0[j]) \neq 0$ for some $j \notin I$, contrary to the assumption. ■

1.2. Denote by $E(\Lambda)$ the Ext-algebra $\bigoplus_{i \geq 0} \text{Ext}_\Lambda^i(\Lambda_0, \Lambda_0)$, with the multiplication given by the Yoneda product. We also consider the even Ext-algebra $E^{\text{ev}}(\Lambda) := \bigoplus_{i \geq 0} \text{Ext}_\Lambda^{2i}(\Lambda_0, \Lambda_0)$,

which is a positively graded algebra with grading $E^{\text{ev}}(\Lambda)_n := \text{Ext}_{\Lambda}^{2n}(\Lambda_0, \Lambda_0)$. For a Λ -module M , let $E(M)$ be the graded $E(\Lambda)$ -module $\bigoplus_{i \geq 0} \text{Ext}_{\Lambda}^i(M, \Lambda_0)$. We also consider the even Ext-module $E^{\text{ev}}(M) := \bigoplus_{n \geq 0} \text{Ext}_{\Lambda}^{2n}(M, \Lambda_0)$ over $E(\Lambda)$, and the odd Ext-module $E^{\text{odd}}(M) := \bigoplus_{n \geq 0} \text{Ext}_{\Lambda}^{2n+1}(M, \Lambda_0)$ over $E^{\text{ev}}(\Lambda)$: they are graded modules with gradings

$$E^{\text{ev}}(M)_n := \text{Ext}_{\Lambda}^{2n}(M, \Lambda_0), \quad \text{and} \quad E^{\text{odd}}(M)_n := \text{Ext}_{\Lambda}^{2n+1}(M, \Lambda_0), \quad \forall n \geq 0.$$

Every graded Λ -module M has a graded projective resolution

$$(1) \quad \mathbf{Q}^{\bullet} : \cdots \rightarrow Q^i \rightarrow \cdots \rightarrow Q^1 \rightarrow Q^0 \rightarrow M \rightarrow 0.$$

If each Q^i is finitely generated, then we say that \mathbf{Q}^{\bullet} is a *finitely generated graded projective resolution* of M . If $M \in \text{gr}(\Lambda)$, then M admits a *minimal* graded projective resolution (1) in the sense that $\text{Im}(Q^i \rightarrow Q^{i-1}) \subseteq JQ^{i-1}$, $\forall i \geq 1$ (see Propositions 2.3 and 2.4 in [GM]).

If $M \in \text{gr}(\Lambda)$, then for each $N \in \text{Gr}(\Lambda)$, $\text{Hom}_{\Lambda}(M, N)$ is a graded k -space with *the shift-grading*: $\text{Hom}_{\Lambda}(M, N)_i = \text{Hom}_{\text{Gr}(\Lambda)}(M, N[i])$, i.e., $\text{Hom}_{\Lambda}(M, N) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\text{Gr}(\Lambda)}(M, N[i])$.

If M has a finitely generated graded projective resolution, then for each $N \in \text{Gr}(\Lambda)$ and each $n \geq 1$, $\text{Ext}_{\Lambda}^n(M, N)$ is a graded k -space with *the shift grading*: $\text{Ext}_{\Lambda}^n(M, N)_i = \text{Ext}_{\text{Gr}(\Lambda)}^n(M, N[i])$, i.e., $\text{Ext}_{\Lambda}^n(M, N) = \bigoplus_{i \in \mathbb{Z}} \text{Ext}_{\text{Gr}(\Lambda)}^n(M, N[i])$.

Fix a minimal graded projective resolution of the trivial Λ -module Λ_0 :

$$(2) \quad \mathbf{P}^{\bullet} : \cdots \rightarrow P^n \rightarrow \cdots \rightarrow P^1 \rightarrow P^0 \rightarrow \Lambda_0 \rightarrow 0.$$

We need the following fact.

Lemma 1.2. ([GMMZ], Lemma 3.2) *Let M be a graded module supported above degree 0 with a minimal graded projective resolution (1). For any integer $n \geq 1$, if P^n in (2) is supported above degree s , then so is Q^n .*

1.3. For the theory of the Koszul algebras and the Koszul modules we refer to A. Beilinson, V. Ginzburg and W. Soergel [BGS], and E. L. Green and R. Martínez-Villa [GM].

Definition 1.3 ([GMMZ], [MM]). *Let $d \geq 2$ be an integer. A graded Λ -module M is a d -Koszul module if M admits a finitely generated graded projective resolution (1) such that each Q^i is generated in degree $\delta(i)$, where*

$$\delta(i) := \begin{cases} nd, & \text{if } i = 2n, \\ nd + 1, & \text{if } i = 2n + 1. \end{cases}$$

If the trivial Λ -module Λ_0 is a d -Koszul module, then we call Λ a d -Koszul algebra.

Theorem 1.4. ([GMMZ], Theorem 6.1) *Let Λ be a d -Koszul algebra and M a d -Koszul Λ -module. Then $E^{\text{ev}}(\Lambda)$ is a Koszul algebra, and $E^{\text{ev}}(M)$ is a Koszul $E^{\text{ev}}(\Lambda)$ -module.*

2. Generalized d -Koszul modules

2.1. Let $d \geq 2$ be an integer. For each integer $i \geq 0$ we assign a subset $\Delta(i)$ of \mathbb{N}_0 as

$$\Delta(i) := \begin{cases} \{nd\}, & \text{if } i = 2n; \\ \{nd+1, \dots, nd+d-1\}, & \text{if } i = 2n+1. \end{cases}$$

Definition 2.1. A graded Λ -module M is called a generalized d -Koszul module if M admits a finitely generated graded projective resolution \mathbf{Q}^\bullet such that each Q^i is generated in degrees in $\Delta(i)$, i.e., $Q^i = \Lambda(\bigoplus_{j \in \Delta(i)} Q_j^i)$, $i \geq 0$.

Remark 2.2. (i) As remarked by Beilinson-Ginzburg-Soergel [BGS] (p.476) in the Koszul situation, \mathbf{Q}^\bullet in Definition 2.1 is unique up to isomorphism.

More precisely, if \mathbf{L}^\bullet is another graded projective resolution of M such that each L^i is also generated in degrees in $\Delta(i)$ (it is not assumed to be finitely generated), then $\mathbf{L}^\bullet \cong \mathbf{Q}^\bullet$ as complexes. In fact, \mathbf{L}^\bullet is homotopy equivalent to \mathbf{Q}^\bullet ; while any chain maps $\mathbf{Q}^\bullet \rightarrow \mathbf{Q}^\bullet$ and $\mathbf{L}^\bullet \rightarrow \mathbf{L}^\bullet$, which respect the grading on Q^i and L^i and are homotopic to zero must themselves be zero (since any element in $\Delta(i)$ is strictly smaller than any element in $\Delta(i+1)$, and Q^i and L^i are both generated in degrees in $\Delta(i)$). It follows that $\mathbf{L}^\bullet \cong \mathbf{Q}^\bullet$ as complexes.

(ii) We emphasize that, as in the d -Koszul situation, here Q^i is also required to be **finitely generated**: it is for the application of the shift grading on $\text{Ext}_\Lambda^n(M, -)$.

(iii) If M is a generalized d -Koszul module, then such a graded projective resolution \mathbf{Q}^\bullet in the definition is minimal, and each syzygy $\Omega^i(M)$ is a graded Λ -module finitely generated in degrees in $\Delta(i)$. In particular, M is finitely generated in degree 0.

(iv) A d -Koszul module is always generalized d -Koszul; and a generalized 2-Koszul module is a finitely generated Koszul module (if Λ is noetherian, then a generalized 2-Koszul Λ -module is exactly a finitely generated Koszul Λ -module).

Example 2.3. Let A be the algebra given by the quiver

$$\alpha \circlearrowleft 1 \xrightarrow{\beta} 2 \xrightarrow{\gamma} 3$$

with relations $\alpha^3, \gamma\beta\alpha$. Then the simple (left) module $S(1)$ has a minimal graded projective resolution

$$\cdots \rightarrow P(1)[4] \oplus P(2)[5] \rightarrow P(1)[3] \oplus P(3)[3] \rightarrow P(1)[1] \oplus P(2)[1] \rightarrow P(1) \rightarrow S(1) \rightarrow 0$$

where $\Omega^4 S(1) = (\Omega^2 S(1))[3]$. Thus $S(1)$ is a generalized 3-Koszul A -module. Since $Q^3 = P(1)[4] \oplus P(2)[5]$ is generated in degrees 4 and 5, but not generated in degree 4, it follows that $S(3)$ is **not** a 3-Koszul A -module (by an argument in Remark 2.2(i)).

2.2. We have the following characterization for a d -Koszul module and for a generalized d -Koszul module, which is the corresponding version of Proposition 2.14.2 in Beilinson - Ginzburg - Soergel [BGS] for the Koszul modules.

Lemma 2.4. *Let M be a graded Λ -module with a finitely generated graded projective resolution. Then*

- (i) *M is d -Koszul if and only if $\text{Ext}_{\Lambda}^i(M, \Lambda_0)_j = \text{Ext}_{\text{Gr}(\Lambda)}^i(M, \Lambda_0[j]) = 0, \forall j \neq \delta(i)$.*
- (ii) *M is generalized d -Koszul if and only if $\text{Ext}_{\Lambda}^i(M, \Lambda_0)$ is concentrated in degrees in $\Delta(i)$, with the shift grading, i.e., $\text{Ext}_{\Lambda}^i(M, \Lambda_0)_j = \text{Ext}_{\text{Gr}(\Lambda)}^i(M, \Lambda_0[j]) = 0, \forall j \notin \Delta(i)$.*

Proof. They can be similarly proved as Proposition 2.14.2 in [BGS]. For the convenience of the reader we include a justification of (ii).

Assume that M is generalized d -Koszul. Then M has a graded projective resolution \mathbf{Q}^{\bullet} such that each Q^i is generated in degrees in $\Delta(i)$, and $\text{Ext}_{\text{Gr}(\Lambda)}^i(M, \Lambda_0[j])$ is the i -th cohomology group of the complex $\text{Hom}_{\text{Gr}(\Lambda)}(\mathbf{Q}^{\bullet}, \Lambda_0[j])$. Since Q^i is generated in degrees in $\Delta(i)$, and $\Lambda_0[j]$ is concentrated in degree j , it follows that $\text{Hom}_{\text{Gr}(\Lambda)}(Q^i, \Lambda_0[j]) = 0$ for $j \notin \Delta(i)$, and hence $\text{Ext}_{\text{Gr}(\Lambda)}^i(M, \Lambda_0[j]) = 0$ for $j \notin \Delta(i)$.

Conversely, assume that $\text{Ext}_{\text{Gr}(\Lambda)}^i(M, \Lambda_0[j]) = 0$ for $j \notin \Delta(i)$. We construct inductively a graded projective resolution \mathbf{L}^{\bullet} of M such that each L^i is generated in degrees in $\Delta(i)$. Since $\text{Hom}_{\text{Gr}(\Lambda)}(M, \Lambda_0[j]) = 0$ for $j \neq 0$, by Lemma 1.1, M is generated in degree 0, and hence we have a surjective graded Λ -homomorphism $L^0 \rightarrow M$ such that L^0 is generated in degree 0. Denote by K^1 its kernel. Then $\text{Hom}_{\text{Gr}(\Lambda)}(K^1, \Lambda_0[j]) = \text{Ext}_{\text{Gr}(\Lambda)}^1(M, \Lambda_0[j]) = 0$ for $j \notin \Delta(1)$, and hence by Lemma 1.1, K^1 is generated in degrees in $\Delta(1)$. Thus we have a surjective graded Λ -homomorphism $L^1 \rightarrow K^1$ such that L^1 is generated in degrees in $\Delta(1)$. Repeating this process we are done.

By assumption we have already a finitely generated graded projective resolution \mathbf{Q}^{\bullet} . By the argument in Remark 2.2(i), there are chain maps $f : \mathbf{L}^{\bullet} \rightarrow \mathbf{Q}^{\bullet}$ and $g : \mathbf{Q}^{\bullet} \rightarrow \mathbf{L}^{\bullet}$ such that $gf = \text{Id}_{\mathbf{L}^{\bullet}}$, which means that \mathbf{L}^{\bullet} is a direct summand of \mathbf{Q}^{\bullet} . Thus \mathbf{L}^{\bullet} is also a finitely generated resolution. By definition M is generalized d -Koszul. \blacksquare

2.3. For a d -Koszul module M , in general $\Omega^i M$ and $J^i M$ are **not** d -Koszul modules, up to shifts (see Proposition 5.2 in [GMMZ] for some special cases); however, they turn out to be generalized d -Koszul, after proper shifts. In the rest of this section we precisely state and prove these results, which will be important in the proof of Main Theorem and Corollary.

Lemma 2.5. *Let M be a d -Koszul Λ -module. Then*

- (i) *$E^{\text{odd}}(M) \cong E^{\text{ev}}(\Omega M)$ as graded $E(\Lambda)$ -modules.*
- (ii) *$(\Omega^i M)[- \delta(i)]$ is a generalized d -Koszul module for each $i \geq 0$.*

Proof. (i) By definition we have an isomorphism of graded $E(\Lambda)$ -modules

$$E^{\text{odd}}(M) = \bigoplus_{n \geq 0} \text{Ext}_{\Lambda}^{2n+1}(M, \Lambda_0) \cong \bigoplus_{n \geq 0} \text{Ext}_{\Lambda}^{2n}(\Omega M, \Lambda_0) = E^{\text{ev}}(\Omega M).$$

(ii) Taking a graded projective resolution \mathbf{Q}^\bullet of M such that each Q^i is finitely generated in degrees in $\delta(i)$, we see that $(\Omega^i M)[- \delta(i)]$ has a graded projective resolution

$$\mathbf{L}^\bullet : \cdots \rightarrow L^j \rightarrow \cdots \rightarrow L^1 \rightarrow L^0 \rightarrow (\Omega^i M)[- \delta(i)] \rightarrow 0$$

where $L^j = Q^{i+j}[- \delta(i)]$ is finitely generated in degree $\delta(i+j) - \delta(i)$ for $j \geq 0$.

If i is even, then L^j is generated in degree $\delta(j)$. That is, $(\Omega^i M)[- \delta(i)]$ is a d -Koszul module, and hence a generalized d -Koszul module.

Assume that i is odd. Then L^j is generated in degree nd if $j = 2n$, and L^j is generated in degree $nd + d - 1$ if $j = 2n + 1$. By definition $(\Omega^i M)[- \delta(i)]$ is generalized d -Koszul. ■

Theorem 2.6. *Let Λ be a d -Koszul algebra and M a generalized d -Koszul Λ -module. Then*

(i) *$(J^i M)[-i]$ is generalized d -Koszul for each $i \geq 1$.*

(ii) *For each $n \geq 1$ we have k -isomorphisms*

$$\mathrm{Ext}_{\mathrm{Gr}(\Lambda)}^{2n-1}(JM, \Lambda_0[nd]) \cong \mathrm{Ext}_{\mathrm{Gr}(\Lambda)}^{2n-1}(J^2 M, \Lambda_0[nd]) \cong \cdots \cong \mathrm{Ext}_{\mathrm{Gr}(\Lambda)}^{2n-1}(J^{d-1} M, \Lambda_0[nd]).$$

Proof. (i) It suffices to prove that $(JM)[-1]$ is generalized d -Koszul. Since $M = \bigoplus_{i \geq 0} M_i$ is finitely generated in degree 0, $JM = \bigoplus_{i \geq 1} M_i$ is finitely generated in degree 1.

We first prove the following claim: $JM[-1]$ admits a graded projective resolution \mathbf{Q}_1^\bullet such that Q_1^i is generated in degrees in $\Delta(i)$. By the proof of Lemma 2.4(ii), it suffices for each $n \geq 0$ to prove that $\mathrm{Ext}_{\mathrm{Gr}(\Lambda)}^{2n}(JM[-1], \Lambda_0[j]) = \mathrm{Ext}_{\mathrm{Gr}(\Lambda)}^{2n}(JM, \Lambda_0[j+1]) = 0$, $\forall j \neq nd$, and that $\mathrm{Ext}_{\mathrm{Gr}(\Lambda)}^{2n+1}(JM, \Lambda_0[j+1]) = 0$, $\forall j \notin \Delta(2n+1) = \{nd+1, \dots, nd+d-1\}$.

Applying $\mathrm{Hom}_{\mathrm{Gr}(\Lambda)}(-, \Lambda_0[j+1])$ to the graded exact sequence $0 \rightarrow JM \rightarrow M \rightarrow M/JM \rightarrow 0$ we get the following exact sequence of k -spaces

$$\begin{aligned} & \mathrm{Ext}_{\mathrm{Gr}(\Lambda)}^{2n}(M, \Lambda_0[j+1]) \rightarrow \mathrm{Ext}_{\mathrm{Gr}(\Lambda)}^{2n}(JM, \Lambda_0[j+1]) \rightarrow \mathrm{Ext}_{\mathrm{Gr}(\Lambda)}^{2n+1}(M/JM, \Lambda_0[j+1]) \\ & \rightarrow \mathrm{Ext}_{\mathrm{Gr}(\Lambda)}^{2n+1}(M, \Lambda_0[j+1]) \rightarrow \mathrm{Ext}_{\mathrm{Gr}(\Lambda)}^{2n+1}(JM, \Lambda_0[j+1]) \rightarrow \mathrm{Ext}_{\mathrm{Gr}(\Lambda)}^{2(n+1)}(M/JM, \Lambda_0[j+1]). \end{aligned}$$

Since Λ is a d -Koszul algebra, P^{2n} in (2) is supported above degrees nd , and hence by Lemma 1.2, Q^{2n} is supported above degrees nd , where \mathbf{Q}^\bullet is a minimal graded projective resolution of $JM[-1]$. Thus $\mathrm{Ext}_{\mathrm{Gr}(\Lambda)}^{2n}(JM, \Lambda_0[j+1]) = 0$ for $j < nd$. Similarly, $\mathrm{Ext}_{\mathrm{Gr}(\Lambda)}^{2n+1}(JM, \Lambda_0[j+1]) = 0$ for $j < nd+1$.

Since M is generalized d -Koszul, by Lemma 2.4(ii), $\mathrm{Ext}_{\mathrm{Gr}(\Lambda)}^{2n}(M, \Lambda_0[j+1]) = 0$ if $j \neq nd-1$, and $\mathrm{Ext}_{\mathrm{Gr}(\Lambda)}^{2n+1}(M, \Lambda_0[j+1]) = 0$ if $j \notin \{nd, \dots, nd+d-2\}$.

Note that M/JM is a Λ/J -module and $\Lambda/J = \Lambda_0$ is a semisimple algebra. Thus M/JM is a direct summand of a finite direct sum of copies of the trivial Λ -module Λ_0 . In particular, M/JM is a d -Koszul module. By Lemma 2.4(i), $\mathrm{Ext}_{\mathrm{Gr}(\Lambda)}^{2n+1}(M/JM, \Lambda_0[j+1]) = 0$ if $j \neq nd$, and $\mathrm{Ext}_{\mathrm{Gr}(\Lambda)}^{2(n+1)}(M/JM, \Lambda_0[j+1]) = 0$ if $j \neq (n+1)d-1$.

Now if $j \neq nd$, then by the exact sequence above we have the following exact sequence

$$\mathrm{Ext}_{\mathrm{Gr}(\Lambda)}^{2n}(M, \Lambda_0[j+1]) \rightarrow \mathrm{Ext}_{\mathrm{Gr}(\Lambda)}^{2n}(JM, \Lambda_0[j+1]) \rightarrow \mathrm{Ext}_{\mathrm{Gr}(\Lambda)}^{2n+1}(M/JM, \Lambda_0[j+1]) = 0,$$

where if $j \neq nd - 1$ then $\text{Ext}_{\text{Gr}(\Lambda)}^{2n}(M, \Lambda_0[j+1]) = 0$, and hence $\text{Ext}_{\text{Gr}(\Lambda)}^{2n}(JM, \Lambda_0[j+1]) = 0$; and if $j = nd - 1 < nd$, then we already know $\text{Ext}_{\text{Gr}(\Lambda)}^{2n}(JM, \Lambda_0[j+1]) = 0$.

Let $j \notin \Delta(2n+1) = \{nd+1, \dots, nd+d-1\}$. Then by the exact sequence above we have the following exact sequence

$$\text{Ext}_{\text{Gr}(\Lambda)}^{2n+1}(M, \Lambda_0[j+1]) \rightarrow \text{Ext}_{\text{Gr}(\Lambda)}^{2n+1}(JM, \Lambda_0[j+1]) \rightarrow \text{Ext}_{\text{Gr}(\Lambda)}^{2(n+1)}(M/JM, \Lambda_0[j+1]) = 0,$$

where if $j \notin \{nd, \dots, nd+d-2\}$ then $\text{Ext}_{\text{Gr}(\Lambda)}^{2n+1}(M, \Lambda_0[j+1]) = 0$, and hence $\text{Ext}_{\text{Gr}(\Lambda)}^{2n+1}(JM, \Lambda_0[j+1]) = 0$; and if $j \in \{nd, \dots, nd+d-2\}$, then $j = nd < nd+1$, and in this case we already know $\text{Ext}_{\text{Gr}(\Lambda)}^{2n+1}(JM, \Lambda_0[j+1]) = 0$. This proves the claim.

Since M/JM is a d -Koszul module, M/JM has a finitely generated graded projective resolution, say \mathbf{Q}_3^\bullet , such that \mathbf{Q}_3^i is generated in degrees in $\delta(i)$. By the graded version of the Horseshoe Lemma, we get a graded projective resolution \mathbf{Q}_2^\bullet of M , such that $\mathbf{Q}_2^i = \mathbf{Q}_1^i \oplus \mathbf{Q}_3^i$ for each i . Thus \mathbf{Q}_2^i is also generated in degrees in $\Delta(i)$. Since M is a generalized d -Koszul module, by Remark 2.2(i), we know that \mathbf{Q}_2^\bullet is finitely generated, and hence \mathbf{Q}_1^\bullet is finitely generated. By definition $JM[-1]$ is generalized d -Koszul.

(ii) Let $d \geq 3$. Applying $\text{Hom}_\Lambda(-, \Lambda_0)$ to the graded exact sequence $0 \rightarrow J^2M \rightarrow JM \rightarrow JM/J^2M \rightarrow 0$, we get the following exact sequence

$$\text{Ext}_\Lambda^{2n-1}(JM/J^2M, \Lambda_0) \rightarrow \text{Ext}_\Lambda^{2n-1}(JM, \Lambda_0) \rightarrow \text{Ext}_\Lambda^{2n-1}(J^2M, \Lambda_0) \rightarrow \text{Ext}_\Lambda^{2n}(JM/J^2M, \Lambda_0).$$

Since $(JM/J^2M)[-1]$ is d -Koszul, by Lemma 2.4(i), $\text{Ext}_{\text{Gr}(\Lambda)}^{2n-1}(JM/J^2M, \Lambda_0[j]) = 0$ if $j \neq nd - d + 2$, and $\text{Ext}_\Lambda^{2n}(JM/J^2M, \Lambda_0[j]) = 0$ if $j \neq nd + 1$. Taking the nd -th homogeneous components of the exact sequence above, we obtain that $\text{Ext}_{\text{Gr}(\Lambda)}^{2n-1}(JM, \Lambda_0[nd]) \cong \text{Ext}_{\text{Gr}(\Lambda)}^{2n-1}(J^2M, \Lambda_0[nd])$. Repeating the process one gets (ii). \blacksquare

3. Proofs of Main Theorem and Corollary

3.1. We begin with a lemma, which seems to be of independent interest.

Lemma 3.1. *Let A be an arbitrary Koszul algebra and \mathcal{C} a full subcategory of $\text{Gr}(A)$. Suppose that for any $X \in \mathcal{C}$, there exist exact sequences in $\text{Gr}(A)$*

$$(3) \quad 0 \rightarrow \Omega \rightarrow P^0 \rightarrow X \rightarrow 0,$$

$$(4) \quad 0 \rightarrow X'' \rightarrow X' \rightarrow \Omega[-1] \rightarrow 0,$$

such that P^0 is a graded projective A -module generated in degree 0 and $X', X'' \in \mathcal{C}$. Then all modules in \mathcal{C} are Koszul A -modules.

Proof. By Proposition 2.14.2 in Beilinson - Ginzburg - Soergel [BGS], it suffices to prove that for each $X \in \mathcal{C}$, $\text{Ext}_{\text{Gr}(A)}^i(X, A_0[j]) = 0$ unless $j = i$. We use induction on i .

The sequence (3) implies that X is generated in degree 0, and hence $\text{Hom}_{\text{Gr}(A)}(X, A_0[j]) = 0$ unless $j = 0$. The sequence (4) implies that Ω is a graded A -module and is generated in degree 1, since $X' \in \mathcal{C}$ is generated in degree 0. By $\text{Ext}_{\text{Gr}(A)}^1(X, A_0[j]) \cong \text{Hom}_{\text{Gr}(A)}(\Omega, A_0[j])$, we see that $\text{Ext}_{\text{Gr}(A)}^1(X, A_0[j]) = 0$ unless $j = 1$.

Let $n \geq 1$. Assume that for each $X \in \mathcal{C}$ and for each positive integer i with $i \leq n$, $\text{Ext}_{\text{Gr}(A)}^i(X, A_0[j]) = 0$ unless $j = i$. The exact sequence (4) implies the following exact sequence for every integer j

$$\text{Ext}_{\text{Gr}(A)}^{n-1}(X'', A_0[j]) \rightarrow \text{Ext}_{\text{Gr}(A)}^n(\Omega[-1], A_0[j]) \rightarrow \text{Ext}_{\text{Gr}(A)}^n(X', A_0[j]).$$

By the inductive hypothesis, we have $\text{Ext}_{\text{Gr}(A)}^{n-1}(X'', A_0[j]) = 0$ unless $j = n-1$, and $\text{Ext}_{\text{Gr}(A)}^n(X', A_0[j]) = 0$ unless $j = n$. Let \mathbf{Q}^\bullet be a minimal graded projective resolution of $\Omega[-1]$ (it exists since $\Omega \subseteq P^0$ is supported above 0). By Lemma 1.2, Q^n is supported above degree n , which implies $\text{Ext}_{\text{Gr}(A)}^n(\Omega[-1], A_0[j]) = 0$ for $j < n$. It follows from the exact sequence above that $\text{Ext}_{\text{Gr}(A)}^n(\Omega[-1], A_0[j]) = 0$ unless $j = n$. Thus $\text{Ext}_{\text{Gr}(A)}^{n+1}(X, A_0[j]) = \text{Ext}_{\text{Gr}(A)}^n(\Omega, A_0[j]) = \text{Ext}_{\text{Gr}(A)}^n(\Omega[-1], A_0[j-1]) = 0$ unless $j = n+1$. This completes the proof. \blacksquare

3.2. Proof of Main Theorem. By Theorem 1.4, $E^{\text{ev}}(\Lambda)$ is a Koszul algebra. Put

$$\mathcal{C} := \{E^{\text{ev}}(N) \in \text{Gr}(E^{\text{ev}}(\Lambda)) \mid N \text{ is a generalized } d\text{-Koszul } \Lambda\text{-module}\}.$$

It suffices to prove that all the conditions in Lemma 3.1 are satisfied.

The graded exact sequence $0 \rightarrow JN \rightarrow N \rightarrow N/JN \rightarrow 0$ induces the following exact sequence of graded k -spaces for each $n \geq 0$

$$(5) \quad \text{Ext}_{\Lambda}^{2n-1}(N, \Lambda_0) \rightarrow \text{Ext}_{\Lambda}^{2n-1}(JN, \Lambda_0) \rightarrow \text{Ext}_{\Lambda}^{2n}(N/JN, \Lambda_0) \rightarrow \text{Ext}_{\Lambda}^{2n}(N, \Lambda_0) \rightarrow \text{Ext}_{\Lambda}^{2n}(JN, \Lambda_0).$$

Since N and $JN[-1]$ are generalized d -Koszul, by Lemma 2.4(ii), we have $\text{Ext}_{\text{Gr}(\Lambda)}^{2n-1}(N, \Lambda_0[nd]) = 0 = \text{Ext}_{\text{Gr}(\Lambda)}^{2n}(JN, \Lambda_0[nd])$. Taking the nd -th homogeneous components of (5) we get the following exact sequence for each $n \geq 0$

$$(6) \quad 0 \rightarrow \text{Ext}_{\text{Gr}(\Lambda)}^{2n-1}(JN, \Lambda_0[nd]) \rightarrow \text{Ext}_{\text{Gr}(\Lambda)}^{2n}(N/JN, \Lambda_0[nd]) \rightarrow \text{Ext}_{\text{Gr}(\Lambda)}^{2n}(N, \Lambda_0[nd]) \rightarrow 0.$$

Since N is generalized d -Koszul and N/JN is d -Koszul, by Lemma 2.4, we have

$$E^{\text{ev}}(N/JN) = \bigoplus_{n \geq 0} \text{Ext}_{\text{Gr}(\Lambda)}^{2n}(N/JN, \Lambda_0[nd]), \quad E^{\text{ev}}(N) = \bigoplus_{n \geq 0} \text{Ext}_{\text{Gr}(\Lambda)}^{2n}(N, \Lambda_0[nd]).$$

By taking direct sum of (6), we get the following short exact sequence in $\text{Gr}(E^{\text{ev}}(\Lambda))$:

$$(7) \quad 0 \rightarrow \Omega \rightarrow E^{\text{ev}}(N/JN) \rightarrow E^{\text{ev}}(N) \rightarrow 0,$$

where $\Omega := \bigoplus_{n \geq 0} \text{Ext}_{\text{Gr}(\Lambda)}^{2n-1}(JN, \Lambda_0[nd])$. In particular, Ω is a graded $E^{\text{ev}}(\Lambda)$ -module with grading $\Omega_n := \text{Ext}_{\text{Gr}(\Lambda)}^{2n-1}(JN, \Lambda_0[nd])$. (One can also prove this directly as follows: since Λ is d -Koszul algebra, it follows from Lemma 2.4(i) that

$$\begin{aligned} \text{Ext}_{\Lambda}^{2m}(\Lambda_0, \Lambda_0) \text{Ext}_{\text{Gr}(\Lambda)}^{2n-1}(JN, \Lambda_0[nd]) &= \text{Ext}_{\text{Gr}(\Lambda)}^{2m}(\Lambda_0, \Lambda_0[md]) \text{Ext}_{\text{Gr}(\Lambda)}^{2n-1}(JN, \Lambda_0[nd]) \\ &\subseteq \text{Ext}_{\text{Gr}(\Lambda)}^{2(n+m)-1}(JN, \Lambda_0[(n+m)d]). \end{aligned}$$

By Theorem 1.4, $E^{\text{ev}}(N/JN)$ is a Koszul $E^{\text{ev}}(\Lambda)$ -module, in particular it is generated in degree 0. Since N/JN is a direct summand of finite direct sum of copies of the trivial Λ -module Λ_0 , $E^{\text{ev}}(N/JN)$ is a projective $E^{\text{ev}}(\Lambda)$ -module.

Similarly, the graded exact sequence $0 \rightarrow J^d N \rightarrow J^{d-1} N \rightarrow J^{d-1} N / J^d N \rightarrow 0$ induces the following exact sequence of graded k -spaces for each $n \geq 0$

$$\begin{aligned} \text{Ext}_{\Lambda}^{2n}(J^{d-1} N, \Lambda_0) &\rightarrow \text{Ext}_{\Lambda}^{2n}(J^d N, \Lambda_0) \rightarrow \text{Ext}_{\Lambda}^{2n+1}(J^{d-1} N / J^d N, \Lambda_0) \\ &\rightarrow \text{Ext}_{\Lambda}^{2n+1}(J^{d-1} N, \Lambda_0) \rightarrow \text{Ext}_{\Lambda}^{2n+1}(J^d N, \Lambda_0). \end{aligned}$$

Note that by Theorem 2.6(i), $J^{d-1} N[-(d-1)]$ and $J^d N[-d]$ are generalized d -Koszul Λ -modules, and that $(J^{d-1} N / J^d N)[-(d-1)]$ is a d -Koszul module. Taking the $(n+1)d$ -th homogeneous components, and by the same arguments we get another exact sequence in $\text{Gr}(\text{E}^{\text{ev}}(\Lambda))$:

$$\begin{aligned} 0 \rightarrow \bigoplus_{n \geq 0} \text{Ext}_{\text{Gr}(\Lambda)}^{2n}(J^d N, \Lambda_0[(n+1)d]) &\rightarrow \bigoplus_{n \geq 0} \text{Ext}_{\text{Gr}(\Lambda)}^{2n+1}(J^{d-1} N / J^d N, \Lambda_0[(n+1)d]) \\ &\rightarrow \bigoplus_{n \geq 0} \text{Ext}_{\text{Gr}(\Lambda)}^{2n+1}(J^{d-1} N, \Lambda_0[(n+1)d]) \rightarrow 0, \end{aligned}$$

or equivalently,

$$(8) \quad 0 \rightarrow \text{E}^{\text{ev}}(J^d N) \rightarrow \text{E}^{\text{odd}}(J^{d-1} N / J^d N) \rightarrow \Omega[-1] \rightarrow 0,$$

where

$$\begin{aligned} \Omega[-1] &= \left(\bigoplus_{n \geq 0} \text{Ext}_{\text{Gr}(\Lambda)}^{2n-1}(J N, \Lambda_0[nd]) \right)[-1] = \bigoplus_{n \geq 0} \text{Ext}_{\text{Gr}(\Lambda)}^{2n+1}(J N, \Lambda_0[(n+1)d]) \\ &\cong \bigoplus_{n \geq 0} \text{Ext}_{\text{Gr}(\Lambda)}^{2n+1}(J^{d-1} N, \Lambda_0[(n+1)d]), \end{aligned}$$

where the last isomorphism follows from Theorem 2.6(ii).

Since $(J^{d-1} N / J^d N)[-(d-1)]$ is d -Koszul, by Lemma 2.5(ii), $\Omega(J^{d-1} N / J^d N)[-d]$ is generalized d -Koszul, and by Lemma 2.5(i), we have

$$\text{E}^{\text{odd}}(J^{d-1} N / J^d N) = \text{E}^{\text{odd}}((J^{d-1} N / J^d N)[-(d-1)]) \cong \text{E}^{\text{ev}}(\Omega(J^{d-1} N / J^d N)[-d]),$$

from which we see $\text{E}^{\text{odd}}(J^{d-1} N / J^d N) \in \mathcal{C}$.

Since N is generalized d -Koszul, by Theorem 2.6(i), $(J^d N)[-d]$ is generalized d -Koszul. Thus $\text{E}^{\text{ev}}(J^d N) = \text{E}^{\text{ev}}((J^d N)[-d]) \in \mathcal{C}$.

Now (7) and (8) shows that all the conditions in Lemma 3.1 are satisfied. This completes the proof. \blacksquare

3.3. Proof of Corollary. By Lemma 2.5(ii), $(\Omega M)[-1]$ is a generalized d -Koszul module. It follows from Main Theorem that $\text{E}^{\text{ev}}((\Omega M)[-1])$ is a Koszul $\text{E}^{\text{ev}}(\Lambda)$ -module. Therefore by Lemma 2.5(i), $\text{E}^{\text{odd}}(M) \cong \text{E}^{\text{ev}}(\Omega M) = \text{E}^{\text{ev}}((\Omega M)[-1])$ is a Koszul $\text{E}^{\text{ev}}(\Lambda)$ -module. \blacksquare

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N. Bian: Dept. Math., Shanghai Jiao Tong University, Shanghai 200240, P. R. China

Dept. Math., Shandong University of Technology, Zibo 255049, P. R. China

Y. Ye: Dept. Math., University of Science and Technology of China, Hefei 230026, P. R. China

P. Zhang: Dept. Math., Shanghai Jiao Tong University, Shanghai 200240, P. R. China